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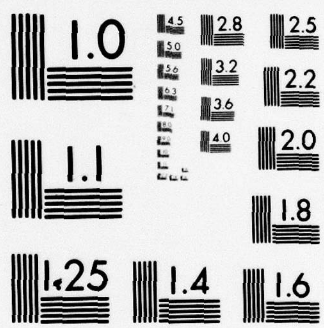

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19 REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER 18 AFOSR-TR-79-1034	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER 9
4. TITLE (and Subtitle) 6 CONVERGENCE OF WEIGHTED SUMS OF RANDOM ELEMENTS IN TYPE P SPACES		5. TYPE OF REPORT & PERIOD COVERED Interim rept.
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) 10 Robert Lee Taylor		8. CONTRACT OR GRANT NUMBER(s) 15 F49620-79-C-0140 17 A5
9. PERFORMING ORGANIZATION NAME AND ADDRESS University of South Carolina Department of Mathematics, Computer Science, and Statistics Columbia, SC 29208		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS 16 61102F 2304/A5 2
11. CONTROLLING OFFICE NAME AND ADDRESS Air Force Office of Scientific Research/NM Bolling AFB, Washington, DC 20332		12. REPORT DATE 11 July 1979
		13. NUMBER OF PAGES 19
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office) 14 TR-43		15. SECURITY CLASS. (of this report) UNCLASSIFIED 12 23
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited

9 Technical rept.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

DDC RECEIVED NOV 30 1979 A

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Weighted sums, random elements, laws of large numbers, convergence, kernel density estimates, and Teoplitz weights.

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Let $\{X_{nk}: n \geq 1, k \geq 1\}$ be an array of row-wise independent random elements in a Banach space of type p, $1 < p \leq 2$. The convergence of $\sum_{k=1}^{\infty} a_{nk} X_{nk}$ in probability and almost surely is obtained under varying moment and distribution conditions on $\{X_{nk}\}$. In particular, laws of large numbers are

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AFOSR-TR- 79-1034

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Convergence of Weighted Sums of Random
Elements in Type p Spaces

by

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Statistics Technical Report No. 43
60B05-13

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[1]Research supported by the Air Force Office of Scientific Research
under Contract Number F49620-79-C-0140.

Abstract

Let $\{X_{nk}: n \geq 1, k \geq 1\}$ be an array of row-wise independent random elements in a Banach space of type p, $1 < p \leq 2$. The convergence of $\sum_{k=1}^{\infty} a_{nk} X_{nk}$ in probability and almost surely is obtained under varying moment and distribution conditions on $\{X_{nk}\}$. In particular, laws of large numbers are obtained for triangular arrays of random elements. Finally, the direct applications of these results in obtaining consistency of the kernel density estimates are indicated.

AMS Classification Number: Primary 60B05; Secondary 60F15.

Key words and phrases: Weighted sums, random elements, laws of large numbers, convergence, kernel density estimates, and Toeplitz weights.

1. Introduction and Preliminaries. The study of laws of large numbers in Banach spaces has led to the geometric considerations of Banach spaces such as Beck convexity (Beck (1963)), G_α (Woycznski (1973)), and type-p (Hoffmann-Jørgensen and Pisier (1976))-as well as many other authors. The study of probability density estimation led to estimates in the form of averages or weighted sums of random variables whose values are in function spaces (Parzen (1962) and Rosenblatt (1971)). As a application of the law of the iterated logarithm in linear measurable spaces, Kuelbs (1978) considered the rates of convergence in these density estimates. The estimates are not always averages of sequences of random elements in a Banach space but are more often weighted sums of arrays of random elements where the weights are not necessarily Toeplitz matrices.

In this paper the convergence of weighted sums of arrays of random elements in Banach spaces of type p is obtained both in probability and almost surely. As corollaries these results have forms of Pruitt's (1963) and Rohatgi's (1971) results for Banach spaces and also have extensions for the results of Padgett and Taylor (1976). However, exact statements of the theorems are related to the possible applications for density estimation. These applications are indicated in Sections 3 and 4.

Let E denote a real separable Banach space with norm $|| \ ||$. Let $(\Omega, \mathcal{A}, P)$ denote a probability space. A random element X in E is a function from Ω into E which is $\mathcal{A}$-measurable with respect to the Borel subsets of E. The expected value of X is defined to be the Pettis integral (when it exists) and is denoted by EX. The moments of a random element X are $E(||X||^p)$ where E is the expected value of the (real-valued) random variable $||X||^p$. The concepts of independence, identical distributions, and convergence have direct extensions to E.

<u>Definition 1</u>: A separable Banach space E is said to be of type p, $1 \le p \le 2$, if and only if there exists $C_1 \in R^+$ such that

$$E(||\sum_{k=1}^{n} X_k||^p) \le C_1 \sum_{k=1}^{n} E(||X_k||^p)$$

for every finite collection of independent random elements $X_1, \ldots, X_n$ in E with mean 0 and finite pth moments.

Hoffmann-Jørgensen and Pisier (1976) had several equivalent statements for type p spaces including the strong law of large numbers. For the results of this paper, the following analogue of Marcinkiewicz-Zygmund inequality by Woyczynski (preprint) will be used.

<u>Proposition 1</u>: Let $1 \le p \le 2$ and $q \ge 1$. Then E is of type p if and only if there exists $C_2 \in R^+$ such that

$$E(||\sum_{k=1}^{n} X_k||^q) \le C_2 E[(\sum_{k=1}^{n} ||X_k||^p)^{q/p}]$$

for all independent random elements $X_1, \ldots, X_n$ with 0 means and finite qth moments.

A collection of random elements $\{X_\alpha\}$ in E is said to be stochastically bounded by a random variable X, $||X_\alpha|| \overset{s}{<} |X|$, if for each α and for each t

$$P[||X_\alpha|| > t] \le P[|X| > t]. \tag{1.1}$$

If the random elements $\{X_\alpha\}$ are identically distributed, then they are stochastically bounded by the random variable $||X_{\alpha_0}||$ for any α_0. Also, by Lemma 5.22 of Taylor (1978) uniformly bounded rth moments (for some $r > 0$) is sufficient for stochastic boundedness by a random variable X and for the existence of $s(s < r)$ moments for the random variable X.

2. Almost Sure Convergence of Weighted Sums. In this section the almost sure convergence for Toeplitz weighted sums of arrays of random elements is obtained. Recall that $\{a_{nk}: n \geq 1, k \geq 1\}$ is a Toeplitz array if

$$\text{(i)} \quad \lim_{n\to\infty} a_{nk} = 0 \text{ for each } k \tag{2.1}$$

$$\text{(ii)} \quad \sum_{k=1}^{\infty} |a_{nk}| \leq \Gamma \text{ for each } n \tag{2.2}$$

where it can be assumed that $\Gamma=1$). When the random elements $\{X_{nk}\}$ in E are stochastically bounded by the random variable X and $E|X| < \infty$, then

$$\sum_{k=1}^{\infty} E||a_{nk}X_{nk}|| \leq \sum_{k=1}^{\infty} |a_{nk}|E|X| \leq E|X|$$

and $\sum_{k=1}^{\infty} a_{nk}X_{nk}$ is convergent almost surely for each n. Theorem 1 will be stated for complete convergence (which implies almost sure convergence) since the proof will consist of showing $\sum_{n=1}^{\infty} P[||\sum_{k=1}^{\infty} a_{nk}X_{nk}|| \geq \varepsilon] < \infty$ for each $\varepsilon > 0$.

Theorem 1: Let $\{X_{nk}\}$ be an array of random elements in a Banach space of type p, $1 < p \leq 2$, which are row-wise independent and such that $EX_{nk} = 0$ for all k and n. If $\{X_{nk}\}$ are stochastically bounded by a random variable X and if $\{a_{nk}\}$ is a Toeplitz array such that $\max_k |a_{nk}| = O(n^{-\gamma})$, $\gamma > 0$, then $E|X|^{1+1/\gamma} < \infty$ implies that

$$||\sum_{k=1}^{\infty} a_{nk}X_{nk}|| \to 0 \qquad \text{completely.}$$

The proof of Theorem 1 is accomplished in three lemmas. The first two lemmas follow directly from Pruitt (1966) and Rohatgi (1971) since $||a_{nk}X_{nk}||=|a_{nk}|\,||X_{nk}||$ and $\{|a_{nk}|\}\&\{||X_{nk}||\}$ are weights and random variables satisfying the corresponding hypotheses.

<u>Lemma 2</u>: If $E|X|^{1+1/\gamma}<\infty$ and $\max_k|a_{nk}|\le Bn^{-\gamma}$, then for every $\varepsilon > 0$

$$\sum_{n=1}^{\infty} P[||a_{nk}X_{nk}|| \ge \epsilon \text{ for some } k] < \infty.$$

<u>Lemma 3</u>: If $E|X|^{1+1/\gamma}<\infty$ and $\max_k|a_{nk}| \le Bn^{-\gamma}$, then for $\alpha < \gamma/2(\gamma+1)$,

$$\sum_{n=1}^{\infty} P[||a_{nk}X_{nk}|| \ge n^{-\alpha} \text{ for at least two values of } k] < \infty.$$

<u>Lemma 4</u>: If $EX_{nk} = 0$ for all n and k, $E|X|^{1+1/\gamma}<\infty$, and $\max_k|a_{nk}| \le Bn^{-\gamma}$, then for every $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} P[||\sum_{k=1}^{\infty} a_{nk}X_{nk}I_{[||a_{nk}X_{nk}||< n^{-\alpha}]}|| \ge \varepsilon] < \infty$$

where $0 < \alpha < \gamma$.

<u>Proof</u>: When $a_{nk} = 0$, then $|a_{nk}|^{-1}$ is understood to be ∞. Since $EX_{nk} = 0$,

$$||E(X_{nk}I_{[||a_{nk}X_{nk}|| < n^{-\alpha}]})||$$

$$=||-E(X_{nk}I_{[||a_{nk}X_{nk}|| \ge n^{-\alpha}]})||$$

$$\le E(||X_{nk}||I_{[||X_{nk}|| \ge n^{-\alpha}|a_{nk}|^{-1}]})$$

$$\le \int_{|x| \ge n^{-\alpha}B^{-1}n^{\gamma}} |x|\, dP[|X| \le x]dx. \tag{2.3}$$

Thus, $\gamma > \alpha$ yields $||E(X_{nk}I_{[||a_{nk}X_{nk}|| < n^{-\alpha}]})|| \to 0$ uniformly in k as $n \to \infty$. Hence,

$$||\sum_{k=1}^{\infty} a_{nk}E(X_{nk}I_{[||a_{nk}X_{nk}|| < n^{-\alpha}]})||$$

$$\le \sum_{k=1}^{\infty} |a_{nk}|E(||X_{nk}||I_{[||a_{nk}X_{nk}|| < n^{-\alpha}]}) \to 0 \tag{2.4}$$

as $n \to \infty$. Define,

$$Z_{nk} = X_{nk}I_{[||a_{nk}X_{nk}|| < n^{-\alpha}]} - E(X_{nk}I_{[||a_{nk}X_{nk}|| < n^{-\alpha}]}). \tag{2.5}$$

Note that $EZ_{nk} = 0$, $E||Z_{nk}||^{1+1/\gamma} \le K_1E||X_{nk}||^{1+1/\gamma} \le K_1E|X|^{1+1/\gamma} = d$ for some $d > 0$, and $||a_{nk}Z_{nk}|| \le 2n^{-\alpha}$. From (2.4)

$$[||\sum_{k=1}^{\infty} a_{nk}X_{nk}I_{[||a_{nk}X_{nk}|| < n^{-\alpha}]}|| > \varepsilon]$$

$$\subset [||\sum_{k=1}^{\infty} a_{nk}Z_{nk}|| \ge \frac{\varepsilon}{2}] \tag{2.6}$$

for sufficiently large n. For each n, pick s(n) so that

$$\sum_{k=s(n)+1}^{\infty} |a_{nk}| \le n^{-2}.$$

Let ν be chosen so that

$$s = \frac{2\nu}{p} \text{ is an integer and } \nu > [(1-\frac{1}{p})(2\gamma)]^{-1}. \tag{2.7}$$

Since $||a_{nk}Z_{nk}|| \le 2n^{-\alpha}$, $E(||\sum_{k=1}^{s(n)} a_{nk}Z_{nk}||^{2\nu})$ is finite. Using Proposition 1

$$E(||\sum_{k=1}^{s(n)} a_{nk}Z_{nk}||^{2\nu}) \le C_2E\ (\sum_{k=1}^{s(n)} ||a_{nk}Z_{nk}||^{p})^{\frac{2\nu}{p}}]$$

$$= C_2 \sum_{k_1,\ldots,k_s} E[\prod_{j=1}^{s} ||a_{nk_j} Z_{nk_j}||^p]. \tag{2.8}$$

In the general term of (2.8), let

$$q_1 \text{ of the k's} = C_1, \quad \ldots \quad , q_m \text{ of the k's} = c_m$$

$$r_1 \text{ of the k's} = b_1, \quad \ldots \quad , r_t \text{ of the k's} = b_t$$

where $p \le pq_i \le 1 + \frac{1}{\gamma}$, $pr_j > 1 + \frac{1}{\gamma}$, and

$$\textstyle\sum_{i=1}^{m} q_i + \sum_{j=1}^{t} r_j = s = \frac{2\nu}{p}.$$

Thus, in (2.8)

$$\begin{aligned}
E[\prod_{j=1}^{s} ||a_{nk_j} Z_{nk_j}||^p] &= E[\prod_{i=1}^{m} ||a_{nc_i} Z_{nc_i}||^{pq_i} \prod_{j=1}^{t} ||a_{nb_j} Z_{nb_j}||^{pr_j}] \\
&= [\prod_{i=1}^{m} |a_{nc_i}|^{pq_i} E(||Z_{nc_i}||^{pq_i})][\prod_{i=1}^{t} E(||a_{nb_j} Z_{nb_j}||^{pr_j})] \\
&\le (1+K_1 E|X|^{1+1/\gamma})^{m+t} (\prod_{i=1}^{m} |a_{nc_i}||a_{nc_i}|^{pq_i-1}) \\
&\qquad \times (\prod_{j=1}^{t} |a_{nb_j}|^{1+1/\gamma})(2n^{-\alpha})^{pr_j - 1 - \frac{1}{\gamma}} \\
&\le (1+d)^{t+m} (\prod_{i=1}^{m} |a_{nc_i}|) (\prod_{j=1}^{t} |a_{nb_j}|)(Bn^{-\gamma})^{\sum_{i=1}^{m}(pq_i-1) + \frac{t}{\gamma}} \\
&\qquad \times (2n^{-\alpha})^{\sum_{j=1}^{t}(pr_j - 1 - \frac{1}{\gamma})}.
\end{aligned} \tag{2.9}$$

For the last expression in (2.9), the power of n^{-1} is

$$\gamma\textstyle\sum_{i=1}^{m}(pq_i-1) + t + \sum_{j=1}^{t}(pr_j - 1 - \frac{1}{\gamma}). \tag{2.10}$$

If $t \ge 1$, then

$$\gamma\textstyle\sum_{i=1}^{m}(pq_i-1) + t + \alpha\sum_{j=1}^{t}(pr_j - 1 - \frac{1}{\gamma})$$

$$\geq 1 + \alpha\sum_{j=1}^{t}(pr_j - 1 - 1/\gamma) \geq 1 + \alpha\delta \tag{2.11}$$

for some fixed $\delta > 0$ since r_j is an integer greater than $\frac{1}{p} + \frac{1}{p\gamma}$. If $t = 0$, then

$$\gamma\sum_{i=1}^{m}(pq_i - 1) + t + \alpha\sum_{j=1}^{t}(pr_j - 1 - \frac{1}{\gamma})$$

$$= \gamma\sum_{i=1}^{m}(pq_i - 1)$$

$$= \gamma(2\nu - \frac{2\nu}{p})$$

$$= \gamma(2\nu)(1 - \frac{1}{p}) > 1$$

by (2.7). By (2.7) and (2.11) $r = \min\{\gamma(2\nu)(1-\frac{1}{p}), 1 + \alpha\delta\} > 1$. From (2.8) and (2.9)

$$E(||\sum_{k=1}^{s(n)+1} a_{nk}Z_{nk}||^{2\nu}) \leq K_2 n^{-r}$$

where the constant K_2 depends only on d, γ, p, and B.

Next,

$$\sum_{n=1}^{\infty} P[||\sum_{k=1}^{\infty} a_{nk}Z_{nk}|| \geq \frac{\varepsilon}{2}]$$

$$\leq \sum_{n=1}^{\infty} P[||\sum_{k=1}^{s(n)+1} a_{nk}Z_{nk}|| \geq \frac{\varepsilon}{4}]$$

$$+ \sum_{n=1}^{\infty} P[||\sum_{k=s(n)+1}^{\infty} a_{nk}Z_{nk}|| \geq \frac{\varepsilon}{4}]$$

$$\leq (\frac{4}{\varepsilon})^{2\nu} \sum_{n=1}^{\infty} E[||\sum_{k=1}^{s(n)+1} a_{nk}Z_{nk}||^{2\nu}]$$

$$+ \frac{4}{\varepsilon}\sum_{n=1}^{\infty}\sum_{k=s(n)+1}^{\infty} E||a_{nk}Z_{nk}||$$

$$\leq (\frac{4}{\varepsilon})^{2\nu} \sum_{n=1}^{\infty} K_2 n^{-r} + \frac{4}{\varepsilon}\sum_{n=1}^{\infty}\sum_{k=s(n)+1}^{\infty} |a_{nk}|(1+d)$$

$$= \left(\frac{4}{\varepsilon}\right)^{2\nu} K_2 \sum_{n=1}^{\infty} n^{-r} + \frac{4}{\varepsilon}(1+d)\sum_{n=1}^{\infty} n^{-2}$$

$$< \infty. \tag{2.12}$$

Finally, (2.6) and (2.12) yield

$$\sum_{n=1}^{\infty} P[||\sum_{k=1}^{\infty} a_{nk} X_{nk}|| \geq \varepsilon] < \infty. \qquad ///$$

The proof of Theorem 1 follows directly from Lemmas 2,3, and 4 since

$$[||\sum_{k=1}^{\infty} a_{nk} X_{nk}|| \geq \varepsilon]$$

$$\subset ([||\sum_{k=1}^{\infty} a_{nk} X_{nk} I_{[||a_{nk} X_{nk}|| < n^{-\alpha}]}|| \geq \frac{\varepsilon}{2}]$$

$$\cup [||a_{nk} X_{nk}|| \geq \frac{\varepsilon}{2} \text{ for some } k]$$

$$\cup [||a_{nk} X_{nk}|| \geq n^{-\alpha} \text{ for at least two } k\text{'s}])$$

A version of Rohatgi's (1968) result for type p, $1 < p \leq 2$, spaces is obtained directly from Theorem 1 as a corollary. It is interesting to observe that the moment condition is related to the weights rather than the type p condition. Typically, $\gamma \leq 1$, and hence $1 + 1/\gamma \geq 2 \geq p$.

<u>Corollary 1</u>: Let $\{X_n\}$ be a sequence of independent random elements in a Banach space of type p, $1 < p \leq 2$, such that $EX_n = 0$ for each n, and let $\{X_n\}$ be stochastically bounded by a random variable X. If $\{a_{nk}\}$ is a Toeplitz array such that $\max_k |a_{nk}| = O(n^{-\gamma})$, $\gamma > 0$, then $E|X|^{1+1/\gamma} < \infty$ implies that

$$||\sum_{k=1}^{\infty} a_{nk} X_k|| \to 0 \text{ completely.}$$

The geometric property of E significantly relaxes the condition imposed in Padgett and Taylor (1976), Theorem 2, to extend Rohatgi's results to Banach spaces. The type p condition is not needed in the corresponding results for independent, identically distributed random elements since the uniform truncation is accomplished by the identical distributions.

3. Convergence in Probability and Density Estimates. Convergence in probability of the weighted sums will be obtained in this section under relaxed conditions on the summability of the weights. The general application of these results to density estimates will be indicated.

Theorem 2: Let $\{X_{nk}\}$ be an array of row-wise independent random elements in a Banach space E of type p, $1 < p \le 2$. Let $EX_{nk} = 0$ for each n and k and let $\{X_{nk}\}$ be stochastically bounded by a random variable X. Let $\{a_{nk}\}$ be an array of constants such that $\max_{1 \le k \le n} |a_{nk}| \to 0$ as $n \to \infty$ and

$$\sum_{k=1}^{\infty} |a_{nk}|^r \le \Gamma \text{ for all } n$$

where $1 \le r < p$. If $E(|X|^r) < \infty$, then

$$\left|\left|\sum_{k=1}^{n} a_{nk} X_{nk}\right|\right| \to 0 \text{ in probability}$$

The proof consists of incorporating the techniques of Theorem 1 into the proof of Theorem 5.3.2 of Taylor (1978), and will be omitted.

In the general density estimation problem $X_1, \ldots, X_n$ are independent random variables with the same density function f. The kernel estimate for f is given by

$$\frac{1}{nh_n} \sum_{k=1}^{n} K\left(\frac{t - X_k}{h_n}\right) = f_n(t) \qquad (3.1)$$

where K is an arbitrarily chosen density and $h_n \to 0$ as $n \to \infty$. There are numerous choices for K and h_n. Here, we will assume that K(t) is a bounded (integrable) kernel with compact support. Let w(t) be a bounded weight function such that

$$\int |f(t)|^p w(t)dt < \infty \tag{3.2}$$

for some $\frac{1}{p} < \lim_n \inf \left[1 + \frac{\log h_n}{\log n}\right]$. Typically, $h_n \sim n^{-\delta}$. Hence, $p > \frac{1}{1-\delta}$ where $0 < \delta < \frac{1}{2}$. Denote

$$E = \{g: g: R \to R \text{ and } \int_{-\infty}^{\infty} |g(t)|^p w(t)dt < \infty\}. \tag{3.3}$$

Then, E is a separable Banach space of type min{2,p} with norm

$$||g|| = (\int |g(t)|^p w(t)dt)^{1/p}. \tag{3.4}$$

Since K is bounded with compact support, $\{K(\frac{t-X_1}{h_n}),\ldots,K(\frac{t-X_n}{h_n})\}$ are i.i.d. random elements in E for each n. For $q \geq 1$

$$\begin{aligned} E(||K(\frac{t-X_1}{h_n})||^q) &= E[(\int_{-\infty}^{\infty} |K(\frac{t-X_1}{h_n})|^p w(t)dt)^{q/p}] \\ &= E[(\int_a^b |K(s)|^p w(X_1+sh_n)h_n ds)^{q/p}] \\ &\leq (\text{bdd } K)^q [(\text{bdd } w)(b-a)h_n]^{q/p} \\ &< \infty. \end{aligned} \tag{3.5}$$

From (3.5) the expected value $E[K(\frac{t-X_1}{h_n})] \in E$, and the random elements $\{K(\frac{t-X_k}{h_n}) - E[K(\frac{t-X_1}{h_n})]: k = 1,2,\ldots,n \text{ and } n \geq 1\}$ are stochastically bounded by a random variable X. Let $a_{nk} = \frac{1}{nh_n}$ for $1 \leq k \leq n$ and $a_{nk} = 0$ for $k > n$. Choose r so that $1 \leq r < p$ and

$$\sum_{k=1}^{\infty} |a_{nk}|^r = n[\frac{1}{nh_n}]^r = n^{1-r} h_n^{-r} \le \Gamma \tag{3.6}$$

for all n. Under the conditions of (3.1) to (3.6), Theorem 2 yields

$$||\frac{1}{nh_n} \sum_{k=1}^{n} (K(\frac{t-X_k}{h_n}) - E(K(\frac{t-X_1}{h_n}))|| \to 0 \tag{3.7}$$

in probability. The troublesome aspect is showing that

$$||\frac{1}{h_n} E(K(\frac{t-X_1}{h_n}))-f(t)|| \to 0. \tag{3.8}$$

<u>Lemma 5</u>: Let f be piecewise continuous, bounded on each compact subset, and $\lim_{|s|\to\infty} f(s)=0$. Then

$$||\frac{1}{h_n} E(K(\frac{t-X_1}{h_n}))-f(t)|| \to 0.$$

<u>Proof</u>: First,

$$||\frac{1}{h_n} E(K(\frac{t-X_1}{h_n}))-f(t)||^p$$

$$= \int_{-\infty}^{\infty} |\frac{1}{h_n} \int_{-\infty}^{\infty} K(\frac{t-s}{h_n}) f(s)ds-f(t)|^p w(t)dt$$

$$= \int_{-\infty}^{\infty} |\int_a^b K(y)[f(t-yh_n)-f(t)]dy|^p w(t)dt. \tag{3.9}$$

The integral with respect to t in (3.9) may need to be broken into several (improper) integrals, so that (w.l.o.g.), it can be assumed that f is continuous at each t. Thus, for each y and each t, $f(t-yh_n) \to f(t)$ as $n \to \infty$. Choose N_1 so that $|yh_n| \le 1$ for all $n \ge N_1$. Hence,

$$|f(t-yh_n)-f(t)| \le |f(t)| + \sup_{-1\le s\le 1} |f(t-s)| \le \Gamma(t)$$

for each t. Hence,

$$\int_a^b K(y)[f(t-yh_n) - f(t)]dy \to 0 \tag{3.10}$$

pointwise in t. Choose α and β so that

$$\int_{-\infty}^{\alpha+1} f(t)dt \le \varepsilon/\text{bdd } w,$$

$$\int_{\beta-1}^{\infty} f(t)dt \le \varepsilon/\text{bdd } w, \text{ and}$$

$$f(t) \le \frac{1}{2} \text{ for } t \notin [\alpha+1,\beta-1]. \tag{3.11}$$

Then for $n \ge N_1$

$$\begin{aligned}
&\int_{-\infty}^{\infty} \Big| \int_a^b K(y)[f(t-yh_n)-f(t)]dy\Big|^p w(t)dt \\
&\le \int_{\infty}^{\alpha} \Big| \int_a^b K(y)[f(t-yh_n)-f(t)]dy\Big| w(t)dt \\
&\quad + \int_{\alpha}^{\beta} \Big| \int_a^b K(y)[f(t-yh_n)-f(t)]dy\Big|^p w(t)dt \\
&\quad + \int_{\beta}^{\infty} \Big| \int_a^b K(y)[f(t-yh_n)-f(t)]\Big| w(t)dt.
\end{aligned} \tag{3.12}$$

The first term of (3.12) becomes

$$\begin{aligned}
&\int_{-\infty}^{\alpha} \Big| \int_a^b K(y)[f(t-yh_n)-f(t)]dy\Big| w(t)dt \\
&\le \text{bdd } w\Big(\int_{-\infty}^{\alpha} \int_a^b K(y)f(t-yh_n)dydt + \int_{-\infty}^{\alpha} \int_a^b K(y)f(t)dydt\Big) \\
&< \varepsilon + \varepsilon
\end{aligned} \tag{3.13}$$

by (3.11) and an interchange of integrals. Similarly, the third term of (3.12) is less than 2ε. The second term of (3.12) converges to zero by the dominated convergence theorem. Hence, there exists N such that

$$\left|\left|\frac{1}{h_n} E(K(\frac{t-X_1}{h_n}))-f(t)\right|\right|^p < 5\varepsilon$$

for all $n \geq N$. ///

Theorem 2 and Lemma 5 provides the consistency

$$||f_n(t)-f(t)|| = \int_{-\infty}^{\infty} |f_n(t)-f(t)|^p w(t)dt \to 0$$

in probability where the pth integrated difference with respect to an arbitrary weight function $w(t)$ is related to the order of h_n. Before proceeding to the complete convergence of the estimates, it is important to note that $w(t) \equiv 1$ is often assumed and that various combinations of conditions on K and f will yield Lemma 5.

4. A Strong Law of Large Numbers Application. In this section a strong law of large numbers is obtained for arrays of random elements in type p spaces. This result substantially improves the density estimation application in the previous section. The proof of Theorem 3 will make use of the following lemma (Corollary 2.1 of Woycznski (pre-print)).

Lemma 6: Let E be of type p, $1 < p \leq 2$, and let $q \geq 1$. If $\{X_n\}$ are independent, identically distributed random elements in E with $E(||X_1||^q) < \infty$ and $EX_1 = 0$, then

$$E(||\textstyle\sum_{k=1}^{n} X_k||^{pq}) \leq C_2 n^q E(||X_1||^{pq}) \tag{4.1}$$

for some constant C_2 independent of n.

Only the major points of the proof of Theorem 3 will be indicated. Again, the proof will use higher moment conditions on the random elements than the type of the space.

Theorem 3: Let $\{X_{nk}\}$ be an array of row-wise independent, identically distributed random elements in a Banach space of type p, $1 < p \le 2$, and let $EX_{nk}=0$ for each n and each k. If for some $q \ge 1$

$$\sum_{n=1}^{\infty} E(||X_{n1}||^{pq})/n^{q(p-1)} < \infty, \tag{4.2}$$

then

$$||\frac{1}{n}\sum_{k=1}^{n} X_{nk}|| \to 0 \text{ completely.}$$

Proof: For each $\varepsilon > 0$ and each n

$$\begin{aligned} P[||\frac{1}{n}\sum_{k=1}^{n} X_{nk}|| > \varepsilon] &\le n^{-pq}\varepsilon^{-pq} E(||\sum_{k=1}^{n} X_{nk}||^{pq}) \\ &\le n^{-pq}\varepsilon^{-pq} C_2 n^q E(||X_{n1}||^{pq}) \end{aligned} \tag{4.3}$$

by Lemma 6. Thus,

$$\begin{aligned} &\sum_{n=1}^{\infty} P[||\frac{1}{n}\sum_{k=1}^{n} X_{nk}|| > \varepsilon] \\ &\qquad \le \varepsilon^{-pq} C_2 \sum_{n=1}^{\infty} E(||X_{n1}||^{pq})/n^{q(p-1)} < \infty \end{aligned}$$

by (4.2). ///

In the density estimation problem, let

$$X_{nk} = \frac{1}{h_n} K(\frac{t-X_k}{h_n}) - \frac{1}{h_n} E(K(\frac{t-X_1}{h_n})).$$

Lemma 5 provides for the convergence

$$\left|\left|\frac{1}{h_n}E(K(\frac{t-X_1}{h_n}))-f(t)\right|\right| \to 0.$$

In verifying the moment condition (4.2), Inequality (3.5) will be used. For any $q \geq 1$

$$\begin{aligned} E(||X_{n1}||^{pq}) &\leq E[(||\frac{1}{h_n}K(\frac{t-X_1}{h_n})|| + ||\frac{1}{h_n}E(K(\frac{t-X_1}{h_n})||)^{pq}] \\ &\leq h_n^{-pq}[E(||K(\frac{t-X_1}{h_n})||^{pq})^{\frac{1}{pq}} + ||E(K(\frac{t-X_1}{h_n}))||]^{pq} \\ &\leq h_n^{-pq}[(C_4h_n^{q})^{\frac{1}{pq}} + (C_5h_n^{\frac{1}{p}})]^{pq} \\ &\leq h_n^{-pq}C_6h_n^{q} = C_6h_n^{q(1-p)} \end{aligned} \tag{4.4}$$

where C_4, C_5, and C_6 are constants depending on the bounds of K and w and the support of K, [a,b], and the values of p and q. Hence,

$$\begin{aligned} E(||X_{n1}||^{pq})/n^{q(p-1)} &\leq C_6h_n^{q(1-p)}/n^{q(p-1)} \\ &= C_6(nh_n)^{-q(p-1)}. \end{aligned} \tag{4.5}$$

Thus, if h_n is of the order $n^{-\delta}$ for any $0 < \delta < 1$ there exists a $q \geq 1$ such that the terms in (4.5) sum. By Theorem 3 and Lemma 5

$$\int|f_n(t)-f(t)|^p w(t)dt \to 0 \text{ completely}$$

for any p, $1 < p \leq 2$, unrelated to the h_n's when

$$\sum_{n=1}^{\infty}(nh_n)^{-\alpha} < \infty$$

for some $\alpha > 0$. Not only is complete convergence obtained, but the h_n's are much more general. This improvement is mainly due to the probability cancellation in type p spaces, and the pth integrated norm (in (4.4)).

Inequality (4.3) suggests an immediate weak law of large numbers for arrays of random elements in type p spaces.

Theorem 4: Let $\{X_{nk}\}$ be an array of row-wise independent, identically distributed random elements in a Banach space of type p, $1 < p \leq 2$, and let $EX_{nk}=0$ for each n and each k. If for some $q \geq 1$

$$n^{-q(p-1)}E(||X_{n1}||^{pq}) \to 0, \tag{4.6}$$

then

$$||\frac{1}{n}\sum_{k=1}^{n} X_{nk}|| \to 0 \text{ in probability.}$$

The condition of row-wise indentically distributed random elements in Theorems 3 and 4 is not necessary since Proposition 1 could be used instead of Lemma 6. However, expressions (4.2) and (4.6) would appear much more complicated.

5. Acknowledgements. Sincere thanks go to Wojbor Woycznski for discussions on the Marcinkiewicz-Zygmund inequality in type p spaces and for suggesting the framework for those results.

REFERENCES

Beck, A. (1963). On the strong law of large numbers. Ergodic Theory. Academic Press, New York, 21-53.

Hoffmann-Jørgensen, J. and Pisier, G. (1976). The law of large numbers and the central limit theorem in Banach spaces. Annals of Probability, 4, 587-599.

Kuelbs, J. (1978), Some exponential moments with applications to density estimation, the empirical distribution function, and Lacunary series. Proceedings of the Conference on Vector Space Measures and Applications. Springer-Verlag Lecture Notes, Vol. 644, 255-269.

Padgett, W. J. and Taylor, R. L. (1976). Almost sure convergence of weighted sums of random elements in Banach spaces. Probability in Banach Spaces, Oberwolfach 1975. Springer-Verlag Lecture Notes, Vol. 526, 187-202.

Parzen E. (1962). On estimation of a probability density and mode. Annals Math. Statistics, 33, 1065-1076.

Pruitt, W. (1966). Summability of independent random variables, J. Math. & Mech., 15, 769-776.

Rohatgi, V. K. (1971). Convergence of weighted sums of independent random variables. Proc. Camb. Phil. Soc., 69, 305-307.

Rosenblatt, M. (1971). Curve estimates. Annals Math Statistics, 42, 1815-1842.

Taylor, R. (1978). Stochastic Convergence of Weighted Sums of Random Elements in Linear Spaces. Springer-Verlag Lecture Notes, Vol. 672.

Wei, D. and Taylor, R. (1978). Geometrical consideration of weighted sums convergence and random weighting. Bull. Inst. of Math., Acad. Sinica, 6, 49-59.

Woycznski, W. (1973). Random series and law of large numbers in some Banach spaces. Theory of Probability and Appl., 18, 350-355.

Woycznski, W. (preprint). On Marcinkiewicz-Zygmund laws of large numbers in Banach spaces and related rates of convergence.

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER AFOSR-TR- 79-1034	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) CONVERGENCE OF WEIGHTED SUM OF RANDOM ELEMENTS IN TYPE P SPACES		5. TYPE OF REPORT & PERIOD COVERED Interim
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Robert Lee Taylor		8. CONTRACT OR GRANT NUMBER(s) F49620-79-C-0140
9. PERFORMING ORGANIZATION NAME AND ADDRESS University of South Carolina Department of Mathematics, Computer Science, and Statistics Columbia, SC 29208		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS 61102F 2304/A5
11. CONTROLLING OFFICE NAME AND ADDRESS Air Force Office of Scientific Research/NM Bolling AFB, Washington, DC 20332		12. REPORT DATE July 1979
		13. NUMBER OF PAGES 19
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Weighted sums, random elements, laws of large numbers, convergence, kernel density estimates, and Teoplitz weights.

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Let $\{X_{nk}: n \geq 1, k \geq 1\}$ be an array of row-wise independent random elements in a Banach space of type p, $1 < p \leq 2$. The convergence of $\sum_{k=1}^{\infty} a_{nk} X_{nk}$ in probability and almost surely is obtained under varying moment and distribution conditions on $\{X_{nk}\}$. In particular, laws of large numbers are

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered) obtained for triangular arrays of random elements. Finally, the direct applications of these results in obtaining consistency of the kerney density estimates are indicated.